

Random phase approximation and extensions applied to a bosonic field theory

H. Hansen^{1,a}, G. Chanfray¹, D. Davesne¹, and P. Schuck²

¹ IPN Lyon, 43 Bd du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France

² IPN Orsay, Université Paris-Sud, France

Received: 18 February 2002
Communicated by V. Vento

Abstract. An application of a self-consistent version of RPA to quantum field theory with broken symmetry is presented. Although our approach can be applied to any bosonic field theory, we specifically study the φ^4 theory in 1 + 1 dimensions. We show that the standard RPA approach leads to an instability which can be removed when going to a superior version, *i.e.* the renormalized RPA. We present a method based on the so-called charging formula of the many-electron problem to calculate the correlation energy and the RPA effective potential.

PACS. 03.70.+k Theory of quantized fields – 21.60.Jz Hartree-Fock and random-phase approximations – 24.10.Cn Many-body theory

1 Introduction

One central aim of the relativistic heavy-ion program is to produce highly excited hot and dense matter possibly constituting a quark-gluon plasma. In such a phase quarks and gluons should be liberated and chiral symmetry should be realized in its Wigner form. A central theoretical question is thus to have a correct description of the broken vacuum and of the progressive restoration of chiral symmetry with increasing temperature and/or baryonic density. This problem is highly non-perturbative in nature and the usual perturbative loop expansion techniques are certainly not sufficient by construction. These features provide at least one important motivation to develop tractable non-perturbative methods to be applied in the context of (effective) field theories with broken symmetries.

One hope is to try to adapt to quantum-field problems very well-controlled non-perturbative methods from the nuclear many-body problem possibly of variational nature. A very popular method, known as the Gaussian approximation for interacting bosons, exactly corresponds to the Hartree-Fock-Bogoliubov mean-field approximation which constitutes the basic building block of the nuclear many-body theory *i.e.* for fermions, see, *e.g.*, [1]. This variational method has already been applied to theories with a global symmetry [2] as well as with a local symmetry [3]. Nevertheless, when this HFB approach is applied to a bosonic $O(N)$ model (*i.e.* to the linear sigma model) the pion ap-

pears with a finite mass or, in other words, Goldstone theorem is violated as discussed in [4]. As demonstrated in recent papers, RPA fluctuations are able to generate this soft mode, known as the spurious mode in the context of nuclear physics [5, 6]. The RPA approach thus appears as a very promising technique to treat non-perturbative problems in the context of quantum field theory. However, in its simplest form, this method is not variational and its predictions are not always very well under control. In particular, it is well known that standard RPA has the tendency to overestimate the attractive correlation energy, at least in examples of nuclear physics. The purpose of this paper is therefore two-fold: first, a complete presentation of the RPA technique is given in the context of the simplest field theory, namely the $\lambda\varphi^4$ theory with a specific application to the 1+1 dimensional case. The second goal is to develop the formalism of more elaborated versions of the RPA approach, namely the so-called renormalized RPA (r-RPA), which, to our knowledge, has never been done before. Let us mention that there exist previous attempts to apply many-body techniques for this specific problem. For instance, Häuser *et al.* use the cluster expansion of Green's functions [7]. Of course other methods, such as lattice calculation, have been employed, see, *e.g.*, [8–10]. Our aim is not really to compete with these numerical methods but actually to develop a tractable approach allowing direct physical interpretations in view of further applications in the context of chiral or gauge theories. In this preliminary work our aim is rather modest and limited to the presentation of the formalism and the discussion of the remaining

^a e-mail: hansen@ipnl.in2p3.fr

problems to be solved such as the explicit covariance of the obtained results. We also present the method to calculate the RPA correlation energy in the context of a field theory within a Green's function formalism. One important point of this paper is to adapt the so-called "charging formula" method (see, *e.g.*, [11]) for quantum field theory beyond the standard RPA scheme. Some numerical results are also obtained showing in particular how to cure the instability of the standard RPA in the broken-symmetry region.

2 The φ^4 theory

2.1 The Hamiltonian

We consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial^\mu \varphi(x) \partial_\mu \varphi(x) - \frac{1}{2} \mu_0^2 \varphi^2(x) - \frac{b}{24} \varphi^4(x), \quad (1)$$

where μ_0^2 is a constant and the bare coupling constant $b = \lambda/6$ is positive for reasons of stability. We decompose the scalar field $\varphi(x)$ in a classical part or condensate s and a fluctuating piece $\phi(x)$:

$$\varphi(x) = \phi(x) + s, \quad s = \langle \varphi(x) \rangle. \quad (2)$$

The presence of the condensate s indicates a spontaneous breaking of the underlying $\varphi \rightarrow -\varphi$ symmetry. Introducing the conjugate field $\Pi(x)$, one obtains for the Hamiltonian (in $d+1$ dimensions)

$$\begin{aligned} H = \int d^d x \left\{ \frac{1}{2} \mu_0^2 s^2 + \frac{b}{24} s^4 + \left(\mu_0 s + \frac{b}{6} s^3 \right) \phi(x) \right. \\ \left. + \frac{1}{2} \left[\Pi^2(x) + (\partial_i \phi)^2(x) + \left(\mu_0^2 + \frac{b}{2} s^2 \right) \phi^2(x) \right] \right. \\ \left. + \frac{b s}{6} \phi^3(x) + \frac{b}{24} \phi^4(x) \right\}. \quad (3) \end{aligned}$$

Putting the system in a large box of volume $V = L^d$, it is convenient to work in momentum space and to expand the fields according to

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{V}} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} \phi_{\vec{q}}(t), \\ \Pi(x) &= -\frac{i}{\sqrt{V}} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} \Pi_{\vec{q}}(t). \quad (4) \end{aligned}$$

The hermiticity of $\phi(x)$ and $\Pi(x)$ imposes $\phi_{\vec{q}}^\dagger = \phi_{-\vec{q}}$ and $\Pi_{\vec{q}}^\dagger = -\Pi_{-\vec{q}}$ and canonical equal-time commutation relations translate into

$$[\phi_{\vec{q}}, \Pi_{\vec{q}'}^\dagger] = \delta_{\vec{q},\vec{q}'}. \quad (5)$$

The Hamiltonian can be rewritten as

$$\begin{aligned} H = V \left(\frac{1}{2} \mu_0^2 s^2 + \frac{b}{24} s^4 \right) \\ + \sqrt{V} \sum_{\vec{q}} \left(\mu_0 s + \frac{b}{6} s^3 \right) \phi_{\vec{q}} \delta_{\vec{q},0} \\ + \sum_{\vec{q}} \left(\Pi_{\vec{q}} \Pi_{\vec{q}}^\dagger + \mathcal{O}_{\vec{q}}^2 \phi_{\vec{q}} \phi_{\vec{q}}^\dagger \right) + \frac{1}{6} \sum_{123} V_{1,2,3} \phi_1 \phi_2 \phi_3 \\ + \frac{1}{24} \sum_{1234} V_{1,2,3,4} \phi_1 \phi_2 \phi_3 \phi_4, \quad (6) \end{aligned}$$

where ϕ_i is a shorthand notation for $\phi_{\vec{q}_i}$ and the three-body and four-body interactions are given by

$$V_{1,2,3} = \frac{b s}{\sqrt{V}} \delta_{\vec{q}_1+\vec{q}_2+\vec{q}_3}, \quad V_{1,2,3,4} = \frac{b}{V} \delta_{\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4}. \quad (7)$$

The bare single-particle energy appearing in the Hamiltonian is

$$\mathcal{O}_{\vec{q}}^2 = \vec{q}^2 + \mu_0^2 + \frac{b}{2} s^2. \quad (8)$$

In the following, we will call H_3 and H_4 the three-body and four-body interacting Hamiltonians.

2.2 The Gaussian approximation

In the Gaussian approximation the ground state is represented by a trial wave function which is a functional of the field $\varphi(\vec{x})$:

$$\begin{aligned} |\psi(\varphi)\rangle &= \\ \mathcal{N} \exp \left(-\frac{1}{2} \int d^d x d^d y (\varphi(\vec{x}) - s) h(\vec{x} - \vec{y}) (\varphi(\vec{y}) - s) \right). \quad (9) \end{aligned}$$

The optimal wave function is obtained by minimizing $\langle \psi(\varphi) | H | \psi(\varphi) \rangle$ with respect to $h(\vec{x} - \vec{y})$. The resulting function, which still depends on the condensate s , defines the Gaussian effective potential. The various minima in s correspond to the possible phases of the system. Working in momentum space, one introduces the Fourier transform of $h(\vec{x} - \vec{y})$ according to

$$h(\vec{x} - \vec{y}) = \int d^d q e^{i\vec{q}\cdot(\vec{x}-\vec{y})} \varepsilon_q. \quad (10)$$

It is easy to show that the trial ground state is the vacuum of the canonical destruction operator $b_{\vec{q}}$ such that

$$\phi_{\vec{q}} = \sqrt{\frac{1}{2\varepsilon_q}} (b_{\vec{q}} + b_{-\vec{q}}^\dagger), \quad \Pi_{\vec{q}} = \sqrt{\frac{\varepsilon_q}{2}} (b_{\vec{q}} - b_{-\vec{q}}^\dagger). \quad (11)$$

The single-particle excitation of this vacuum have energies ε_q which differ from the bare energies \mathcal{O}_q . In other words we have rotated the original bare basis with single-particle energies \mathcal{O}_q into a basis associated to the ε_q 's through a Hartree-Fock-Bogoliubov (HFB) transformation. In this

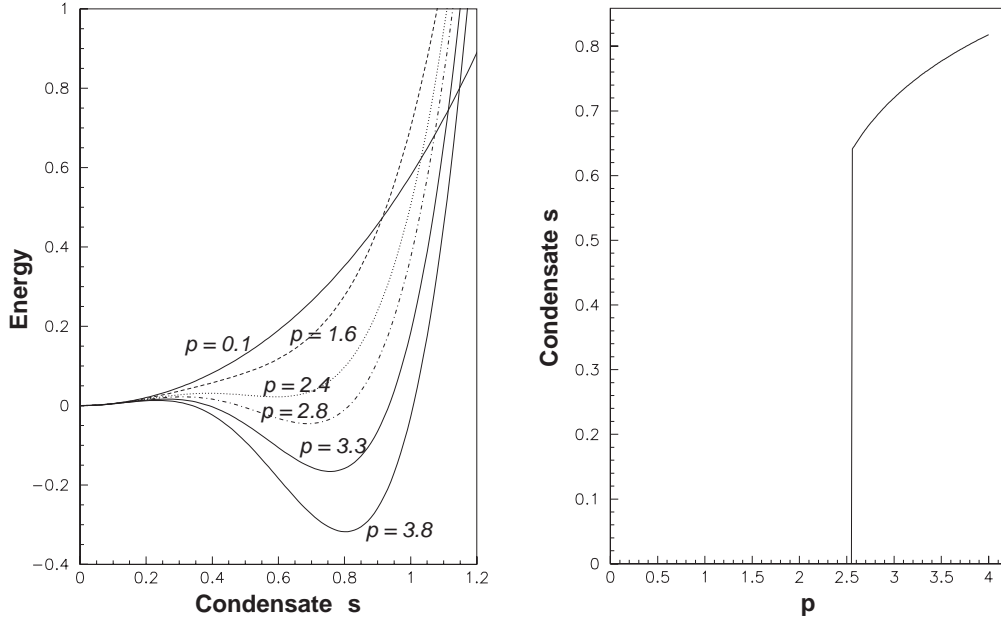


Fig. 1. Left panel: Gaussian effective potential for various values of the dimensionless coupling constant p . Right panel: value of the condensate s minimizing the Gaussian effective potential as a function of p .

HFB ground state the energy density is easily calculated with by using the Wick theorem

$$\mathcal{E}_0(\varepsilon, s) = \frac{1}{2}\mu_0^2 s^2 + \frac{b}{24} s^4 + \frac{1}{2V} \sum_{\vec{q}} \left(\frac{\varepsilon_{\vec{q}}}{2} + \frac{\mathcal{O}_{\vec{q}}^2}{2\varepsilon_{\vec{q}}} \right) + \frac{b}{8} \langle \phi^2 \rangle^2, \quad (12)$$

where the scalar density $\langle \phi^2 \rangle$ is given by

$$\langle \phi^2 \rangle = \frac{1}{V} \sum_{\vec{q}} \langle \phi_{\vec{q}} \phi_{\vec{q}}^\dagger \rangle = \frac{1}{V} \sum_{\vec{q}} \frac{1}{2\varepsilon_{\vec{q}}} \equiv \int \frac{d^d q}{(2\pi)^d} \frac{1}{2\varepsilon_{\vec{q}}}. \quad (13)$$

Minimization with respect to $\varepsilon_{\vec{q}}$ gives the HFB quasi-particle mass m :

$$m^2 = \varepsilon_{\vec{q}}^2 - \vec{q}^2 = \mu_0^2 + \frac{b}{2} (s^2 + \langle \phi^2 \rangle). \quad (14)$$

It has been demonstrated in $d = 3$ spatial dimensions that the above gap equation can be rendered finite by appropriate mass and coupling-constant renormalizations [12]. Here we concentrate onto the $d = 1$ case where the theory becomes super-renormalizable and only requires a mass renormalization. We eliminate μ_0 in favor of the renormalized mass μ according to

$$\mu_0^2 = \mu^2 - \frac{b}{2} \int_{-\Lambda}^{+\Lambda} \frac{dq}{2\pi} \frac{1}{2\sqrt{q^2 + \mu^2}}, \quad (15)$$

where Λ is a ultraviolet cutoff. The gap equation becomes

$$m^2 = \mu^2 + \frac{b}{2} (s^2 + \langle \Delta \phi^2 \rangle_{\mu}) \quad (16)$$

$$\langle \Delta \phi^2 \rangle_{\mu} = \int_{-\Lambda}^{+\Lambda} \frac{dq}{2\pi} \left(\frac{1}{2\sqrt{q^2 + m^2}} - \frac{1}{2\sqrt{q^2 + \mu^2}} \right) - \frac{1}{4\pi} \ln \frac{m^2}{\mu^2} \quad (17)$$

which is independent of the cutoff Λ . Re-injecting the solution for the mass m into the expression of the energy density one gets the effective potential which is also finite, once the energy of the perturbative vacuum of particles with mass μ is removed. The result is

$$\frac{\mathcal{E}_0(s)}{\mu^2} = \frac{1}{2} s^2 + p s^4 + \frac{1}{8\pi} \left(\frac{m^2}{\mu^2} - 1 - \frac{m^2}{\mu^2} \ln \frac{m^2}{\mu^2} \right) - \frac{3p}{16\pi^2} \left(\ln \frac{m^2}{\mu^2} \right)^2, \quad (18)$$

where

$$p = b/24\mu^2$$

is the dimensionless coupling constant. At low p the effective potential has a minimum at $s = 0$: the unbroken phase is thus stable. Increasing p , a new minimum develops at finite s corresponding to a meta-stable broken phase. At a certain critical $p_c = 2.57$ the deformed phase becomes stable and the system undergoes a first-order phase transition (see fig. 1) in agreement with [7]. The approach can be straightforwardly extended to finite temperature using the so-called statistical variational principle.

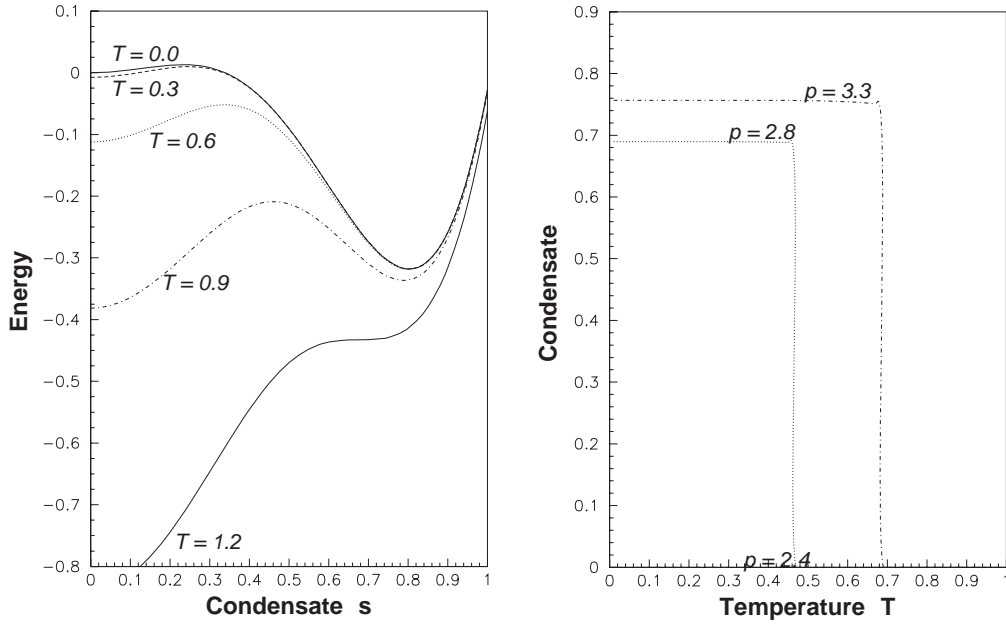


Fig. 2. Left panel: Gaussian effective potential for $p = 3.8$ for various temperatures T in units of μ . Right panel: value of the condensate s minimizing the Gaussian effective potential as a function of T (in units of μ) for various values of p .

One obtains for the finite-temperature effective potential (*i.e.* the grand potential)

$$\frac{\Omega_0(s, T)}{\mu^2} = \frac{T}{\mu} \int \frac{dq}{2\pi} \ln \left(1 - \exp \left(\frac{\varepsilon_q}{T} \right) \right) + \frac{1}{2} s^2 + p s^4 + \frac{1}{8\pi} \left(\frac{m^2}{\mu^2} - 1 - \frac{m^2}{\mu^2} \ln \frac{m^2}{\mu^2} \right) - 3p \langle \Delta\phi^2 \rangle_\mu^2(T) \quad (19)$$

with

$$\langle \Delta\phi^2 \rangle_\mu(T) = \int \frac{dq}{2\pi} \left(\frac{1 + n(\varepsilon_q/T)}{2\sqrt{q^2 + m^2}} - \frac{1}{2\sqrt{q^2 + \mu^2}} \right), \quad (20)$$

where $n(x) = 1/(\exp(x) - 1)$ is the Bose-Einstein distribution and the gap equation (16) is unchanged once $\langle \Delta\phi^2 \rangle_\mu$ is replaced by its finite-temperature expression given just above. Some numerical results are shown on fig. 2. Choosing $p > p_c$ the vacuum is in a broken phase. At a certain critical temperature T_c one gets a first-order transition towards the symmetry restored phase.

3 The RPA approach

3.1 The equation-of-motion method (EOM)

The aim of the RPA method is to describe the excitation spectrum of a Hamiltonian H . The excited states $|\nu\rangle$ and the ground state $|0\rangle$ are defined by the conditions

$$|\nu\rangle = Q_\nu^\dagger |0\rangle, \quad Q_\nu |0\rangle = 0. \quad (21)$$

Minimizing $E_\nu = \langle \nu | H | \nu \rangle / \langle \nu | \nu \rangle$ with respect to the operators Q_ν , one gets the following set of equations [1, 13]:

$$\langle 0 | [\delta Q_\nu, [H, Q_\nu^\dagger]] | 0 \rangle = \Omega_\nu \langle 0 | [\delta Q_\nu, Q_\nu^\dagger] | 0 \rangle, \quad (22)$$

where $\Omega_\nu = E_\nu - E_0$ is the excitation energy. Equation (22) is supplemented by the conditions

$$\langle 0 | [H, Q_\nu] | 0 \rangle = 0 \quad (23)$$

which will determine the mean-field basis. Equation (23) is a natural complement to eq. (22), since both equations hold in the exact case [14]. In general in this EOM approach, the excitation operators are searched only within a limited domain. They are usually taken in the form

$$Q_\nu^\dagger = \sum_a (X_a^\nu A_a^\dagger - Y_a^\nu A_{-a}), \quad (24)$$

where the A_a constitute a set of non-Hermitian operators labeled by a set of quantum numbers a (for instance momentum or isospin state). The RPA equations (22) now become matrix equations which allow to determine the excitation energy and the X and Y amplitudes (summation over repeated indices is understood):

$$\begin{pmatrix} \mathcal{A}_{ab} & \mathcal{B}_{ab} \\ \mathcal{B}_{-a-b}^* & \mathcal{A}_{-a-b}^* \end{pmatrix} \begin{pmatrix} X_b^\nu \\ Y_b^\nu \end{pmatrix} = \Omega_\nu \mathcal{N}_{ab} \begin{pmatrix} X_b^\nu \\ Y_b^\nu \end{pmatrix} \quad (25)$$

with the matrix elements given by the double commutators

$$\begin{aligned} \mathcal{A}_{ab} &= \langle 0 | [A_a, [H, A_b^\dagger]] | 0 \rangle, \\ \mathcal{B}_{ab} &= -\langle 0 | [A_a, [H, A_{-b}]] | 0 \rangle \end{aligned} \quad (26)$$

and the norm matrix is

$$\mathcal{N}_{ab} = \begin{pmatrix} \langle 0 | [A_a, A_b^\dagger] | 0 \rangle & \langle 0 | [A_a, A_{-b}] | 0 \rangle \\ -\langle 0 | [A_a, A_{-b}] | 0 \rangle & -\langle 0 | [A_a, A_b^\dagger] | 0 \rangle \end{pmatrix}. \quad (27)$$

The previous conditions (23) translate into equations having the form of gap equations:

$$\langle 0 | [H, A_a] | 0 \rangle = 0. \quad (28)$$

In practice the quality of the RPA scheme depends on the approximation which are made. Firstly the larger is the space of A_a operators, the better is the accuracy of the method. Here, in practice we will limit ourselves to single- and pair-boson operators namely $A_a^\dagger = \{b_{\vec{q}}^\dagger, b_{\vec{q}}^\dagger b_{\vec{q}'}^\dagger, b_{\vec{q}}^\dagger b_{\vec{q}'}^\dagger\}$. Secondly, one crucial point is the calculation of the ground-state matrix elements of the double commutators and of the matrix elements appearing in the gap equations. Calculating the matrix elements on the true RPA ground state constitutes the self-consistent RPA (SCRPA). In practice this is a formidable task both formally and numerically which only has been fully achieved in a very limited number of simple model cases. In the most common approximation (*i.e.* the standard RPA), the ground-state matrix elements entering the RPA equations are calculated on the mean-field ground state. In our case it coincides with the Gaussian or HFB ground state discussed above. There is an intermediate approximation scheme, called in nuclear physics renormalized RPA (r-RPA)[13,15–17], which allows to incorporate part of the correlations in the ground state. In sect. 4, we will propose the first application of this method in the context of quantum field theory.

3.2 The Dyson equation approach (DEA)

We present an alternative but equivalent formulation of the RPA scheme based on a Green's function (GF) approach [18]. Although equivalent in principle to the equation-of-motion approach, this DEA method is more convenient for quantum field theory. We define the time-ordered Green's functions:

$$\begin{aligned} G_{a,\bar{b}}(t, t') &= -i \langle 0 | T (A_a(t), A_{\bar{b}}^\dagger(t')) | 0 \rangle, \\ G_{a,-b}(t, t') &= -i \langle 0 | T (A_a(t), A_{-b}(t')) | 0 \rangle, \\ G_{-\bar{a},\bar{b}}(t, t') &= -i \langle 0 | T (A_{-\bar{a}}^\dagger(t), A_{\bar{b}}^\dagger(t')) | 0 \rangle, \\ G_{-\bar{a},-b}(t, t') &= -i \langle 0 | T (A_{-\bar{a}}^\dagger(t), A_{-b}(t')) | 0 \rangle. \end{aligned} \quad (29)$$

We introduce the energy representation of these GF according to

$$G(t, t') = \int \frac{dE}{2\pi} e^{-iE(t-t')} G(E) \quad (30)$$

and the matrix

$$\mathcal{G}_{a,b}(E) = \begin{pmatrix} G_{a,\bar{b}}(E), & G_{a,-b}(E) \\ G_{-\bar{a},\bar{b}}(E), & G_{-\bar{a},-b}(E) \end{pmatrix}. \quad (31)$$

The RPA equations have now the form of a set of coupled integral equations for the various GF:

$$E \mathcal{G}_{a,b}(E) = \mathcal{N}_{a,b} + \sum_{c,d} \begin{pmatrix} \mathcal{A}_{ac}, & \mathcal{B}_{ac} \\ \mathcal{B}_{-a-c}^*, & \mathcal{A}_{-a-c}^* \end{pmatrix} \mathcal{N}_{c,d}^{-1} \mathcal{G}_{d,b}(E) \quad (32)$$

with

$$\begin{aligned} \mathcal{A}_{ab} &= \langle 0 | [[A_a, H], A_b^\dagger] | 0 \rangle, \\ \mathcal{B}_{ab} &= -\langle 0 | [[A_a, H], A_{-b}] | 0 \rangle. \end{aligned} \quad (33)$$

Notice that the double commutators are ordered differently from the ones obtained in the equation-of-motion method. In practice, it can be shown that they are always identical.

4 The RPA and the renormalized RPA applied to the φ^4 theory

4.1 Solution of the RPA problem

We will limit ourselves to the case where the A operators are only one-body and two-body operators. We introduce creation b_β^\dagger and destruction operators b_β depending on the parameters κ_β according to

$$b_\beta = \sqrt{\frac{\kappa_\beta}{2}} \phi_\beta + \sqrt{\frac{1}{2\kappa_\beta}} \Pi_\beta, \quad (34)$$

or equivalently

$$\phi_\beta = \sqrt{\frac{1}{2\kappa_\beta}} (b_\beta + b_{-\beta}^\dagger), \quad \Pi_\beta = \sqrt{\frac{\kappa_\beta}{2}} (b_\beta - b_{-\beta}^\dagger). \quad (35)$$

By construction they obey standard canonical commutation relation for boson creation and destruction operators:

$$[b_\beta, b_{\beta'}^\dagger] = \delta_{\beta,\beta'}, \quad [b_\beta, b_{\beta'}] = 0. \quad (36)$$

The β 's represent the quantum numbers of the created boson state; here this is simply a momentum index and $-\beta$ represents the opposite momentum.

The excitation operators A_a^\dagger will be the one-body operators b_α^\dagger and the two-body operators $b_\beta^\dagger b_{\beta'}^\dagger$ and $b_\beta^\dagger b_{-\beta'}^\dagger$ with $\beta \neq \beta'$ for the ‘‘particle-hole’’ operators. The gap equation $\langle [H, b_\beta] \rangle = 0$ will give the extrema in the condensate s of the vacuum energy. Since we will calculate the effective potential giving directly the true minimum (*i.e.* the stable phase) we will not use it here. The most interesting non-trivial gap equation is $\langle [H, b_\beta b_{-\beta}] \rangle = 0$ which constrains the basis, that is the κ_β parameters. This gap equation gives ($\langle \dots \rangle$ stands for the ground-state expectation value)

$$\begin{aligned} \langle \Pi_\beta \Pi_\beta^\dagger \rangle - \varepsilon_\beta^2 \langle \phi_\beta \phi_\beta^\dagger \rangle &= \frac{1}{2} \sum_{1,2} \langle \phi_\beta \phi_1^\dagger \phi_2^\dagger \rangle_{\text{conn}} V_{1,2,-\beta} \\ &+ \frac{1}{6} \sum_{1,2,3} \langle \phi_\beta \phi_1^\dagger \phi_2^\dagger \phi_3^\dagger \rangle_{\text{conn}} V_{1,2,3,-\beta}, \end{aligned} \quad (37)$$

where the suffix ‘‘conn’’ means connected operators which contain exclusively correlated expectation values:

$$\begin{aligned} \langle ABCD \rangle_{\text{conn}} &= \langle ABCD \rangle - \langle AB \rangle \langle CD \rangle \\ &- \langle AC \rangle \langle BD \rangle - \langle AD \rangle \langle BC \rangle \end{aligned} \quad (38)$$

and ε_β is given by

$$\varepsilon_\beta^2 = \bar{q}_\beta^2 + \mu_0^2 + \frac{b}{2}s^2 + \frac{b}{2}\langle\phi^2\rangle \quad (39)$$

with

$$\begin{aligned} \langle\phi^2\rangle &= \frac{1}{V} \sum_{\bar{q}} \langle\phi_{\bar{q}}\phi_{\bar{q}}^\dagger\rangle = \\ &= \frac{1}{V} \sum_{\bar{q}} \frac{1}{2\kappa_{\bar{q}}} \langle 1 + 2b_{\bar{q}}^\dagger b_{\bar{q}} + b_{\bar{q}}^\dagger b_{-\bar{q}}^\dagger + b_{\bar{q}} b_{-\bar{q}} \rangle. \end{aligned} \quad (40)$$

The ε_β 's can be seen as the generalized mean-field single-particle energies. But at variance with the Gaussian case these energies depend on the correlated self-consistent scalar density $\langle\phi^2\rangle$.

Standard RPA

In the standard RPA, we omit the connected parts. We thus get from the gap equation (37)

$$\langle\Pi_\beta\Pi_\beta^\dagger\rangle - \varepsilon_\beta^2 \langle\phi_\beta\phi_\beta^\dagger\rangle = 0 \quad (41)$$

and take for $\langle\phi_\beta\phi_\beta^\dagger\rangle$ its value in the Gaussian HFB ground state. This implies (see eq. (35))

$$\kappa_\beta = \varepsilon_\beta. \quad (42)$$

The basis is fixed and coincides with the HFB basis. Similarly all the matrix elements appearing in the calculation of the double commutators and the norm matrix are simply obtained by using the Wick theorem on the Gaussian ground state.

Renormalized RPA

In renormalized RPA (r-RPA), one still systematically omits the correlated expectation values. Hence, the gap equation (41) remains valid. However, the scalar density is not yet fixed. It has to be determined self-consistently. The problem of its evaluation (in a non-relativistic many-body problem for Fermi systems it corresponds to the occupation numbers) is one of the subtleties of self-consistent RPA (SCRPA). We will come to this problem later on. The gap equation can be rewritten as

$$(\kappa_\beta^2 - \varepsilon_\beta^2) \langle 1 + 2b_\beta^\dagger b_\beta \rangle = (\kappa_\beta^2 + \varepsilon_\beta^2) \langle b_\beta^\dagger b_{-\beta}^\dagger + b_\beta b_{-\beta} \rangle. \quad (43)$$

The basis κ_β is not yet totally fixed. However, we have checked that the result of the r-RPA calculation does not depend on its choice, provided the above gap equation is satisfied. Thus, we can choose it as $\kappa_\beta = \varepsilon_\beta$, *i.e.* the generalized mean-field basis. This choice has the merit of significantly simplifying the lengthy calculation of the double commutators. In that case one has

$$\langle b_\beta^\dagger b_{-\beta}^\dagger \rangle = \langle b_\beta b_{-\beta} \rangle = 0. \quad (44)$$

In SCRPA this last property has to be always satisfied because the $b_\beta^\dagger b_{-\beta}^\dagger$ operators are just linear combinations of the genuine RPA excitation operators Q_ν^\dagger whose expectation values on the true RPA ground state vanish by construction.

The many-body operator expectation values are also calculated using the Wick theorem but the resulting two-body operator matrix elements depend on the various occupation numbers. In other words all the ground-state matrix elements entering the double commutators and norm matrices are expressible in term of the fixed momentum densities $\mathcal{N}_\beta = \langle\phi_\beta\phi_\beta^\dagger\rangle$. We display here some examples:

$$\begin{aligned} \langle b_\beta b_{-\beta} \rangle &= \frac{\kappa_\beta^2 - \varepsilon_\beta^2}{2\varepsilon_\beta} \mathcal{N}_\beta, \\ \langle b_\beta^\dagger b_\beta \rangle &= -\frac{1}{2} + \frac{\kappa_\beta^2 + \varepsilon_\beta^2}{2\varepsilon_\beta} \mathcal{N}_\beta, \\ \langle b_1^\dagger b_2^\dagger b_3 b_4 \rangle &= \langle b_1^\dagger b_1 \rangle \langle b_2^\dagger b_2 \rangle (\delta_{1,3} \delta_{2,4} + \delta_{1,4} \delta_{2,3}) \\ &\quad + \langle b_1^\dagger b_{-1}^\dagger \rangle \langle b_3 b_{-3} \rangle (\delta_{1,-2} \delta_{3,-4}). \end{aligned} \quad (45)$$

We do not give the details of the calculation to obtain the solution of the RPA problem (32). Even if we limit ourselves to one- and two-body operators, it represents quite a lot of algebra. The results for the various GF are listed in the appendix. Here we give only the GF relative to the field operator $\phi_{\vec{P}}$. For the one-particle GF, *i.e.* the ϕ particle propagator, one obtains

$$\begin{aligned} G_{\phi_{\vec{P}},\phi_{\vec{P}'}}(E) &\equiv \delta_{\vec{P},\vec{P}'} G(E,\vec{P}), \\ G(E,\vec{P}) &= \left(E^2 - \varepsilon_{\vec{P}}^2 - \Sigma(E,\vec{P}) \right)^{-1}, \end{aligned} \quad (46)$$

where the mass operator has the following form:

$$\Sigma(E,\vec{P}) = \frac{b^2 s^2}{2} \tilde{I}(E,\vec{P}),$$

with

$$\tilde{I}(E,\vec{P}) = \frac{I(E,\vec{P})}{1 - \frac{b}{2}I(E,\vec{P})} \quad (47)$$

and the two-particle loop has the explicit expression

$$\begin{aligned} I(E,\vec{P}) &= \int \frac{d\vec{k}_1 d\vec{k}_2}{(2\pi)^d} \delta^{(d)}(\vec{P} - \vec{k}_1 - \vec{k}_2) \\ &\quad \times \left[\frac{\varepsilon_1 + \varepsilon_2}{2\varepsilon_1\varepsilon_2} \frac{\varepsilon_1 \mathcal{N}_1 + \varepsilon_2 \mathcal{N}_2}{E^2 - (\varepsilon_1 + \varepsilon_2)^2 + i\eta} \right. \\ &\quad \left. - \frac{\varepsilon_1 - \varepsilon_2}{2\varepsilon_1\varepsilon_2} \frac{\varepsilon_1 \mathcal{N}_1 - \varepsilon_2 \mathcal{N}_2}{E^2 - (\varepsilon_1 - \varepsilon_2)^2 + i\eta} \right]. \end{aligned} \quad (48)$$

Notice that correlations are present in this expression through the densities $\mathcal{N}_i = \langle\phi_i\phi_i^\dagger\rangle$. We will see in subsect. 4.4 how to calculate these densities in r-RPA. For the 1p-2p and 2p-2p GF we give the particular combinations which are directly relevant for the calculation of the

$$\begin{aligned} & \frac{1}{24V} \int \frac{i dE}{(2\pi)} e^{iE\eta^+} \left[\sum_{1234} V_{1,2,-3,-4} (G_{\phi_3\phi_4, \phi_1^\dagger\phi_2^\dagger}(E) - G_{\phi_3\phi_4, \phi_1^\dagger\phi_2^\dagger}^0) \right] = \\ & \frac{1}{24} \int \frac{d\vec{P}}{(2\pi)^d} \int \frac{i dE}{(2\pi)} e^{iE\eta^+} \left[b^2 \frac{I^2(E, \vec{P})}{1 - \frac{b}{2} I(E, \vec{P})} + b^3 s^2 \left(\frac{I(E, \vec{P})}{1 - \frac{b}{2} I(E, \vec{P})} \right)^2 G(E, \vec{P}) \right] \end{aligned} \quad (51)$$

effective potential

$$\begin{aligned} & \sum_{123} V_{1,-2,-3} G_{\phi_2\phi_3, \phi_1^\dagger}(E) = \\ & V \int \frac{d\vec{P}}{(2\pi)^d} 2 \Sigma(E, \vec{P}) G(E, \vec{P}), \\ & \sum_{1234} V_{1,2,-3,-4} G_{\phi_3\phi_4, \phi_1^\dagger\phi_2^\dagger}(E) = \\ & V \int \frac{d\vec{P}}{(2\pi)^d} b \left[2I(E, \vec{P}) + b \frac{I^2(E, \vec{P})}{1 - \frac{b}{2} I(E, \vec{P})} \right. \\ & \left. + b^2 s^2 \left(\frac{I(E, \vec{P})}{1 - \frac{b}{2} I(E, \vec{P})} \right)^2 G(E, \vec{P}) \right]. \end{aligned} \quad (49)$$

The reader may check that these results have a very clear diagrammatic interpretation (see fig. 4 in the following). From these expressions, one can get the expectation value of the three-body Hamiltonian by taking the appropriate $t' \rightarrow t$ limit of the GF

$$\langle H_3 \rangle / V = \frac{1}{6} \int \frac{d\vec{P}}{(2\pi)^d} \int \frac{i dE}{(2\pi)} e^{iE\eta^+} 2 \Sigma(E, \vec{P}) G(E, \vec{P}). \quad (50)$$

The correlated part of the four-body Hamiltonian is obtained with the same technique but the uncorrelated GF has to be removed. As we will see below, the expression of the correlated energy will involve the following quantity:

see eq. (51) above

where the first-order term (*i.e.* the one-loop term $I(E, \vec{P})$ of eq. (49)) has been removed.

4.2 The effective potential

The starting Hamiltonian equations (6), (7) contains a free Hamiltonian of bare particles with energy $\mathcal{O}_{\vec{q}}$ and an interacting Hamiltonian $H_3 + H_4$. To simplify the writing we now replace the momentum labels \vec{q} by integers i and omit the linear term in ϕ which does not directly play a role in the formal manipulations:

$$\begin{aligned} H = & V \left(\frac{1}{2} \mu_0^2 s^2 + \frac{b}{24} s^4 \right) \\ & + \sum_1 \frac{1}{2} \left(\Pi_1 \Pi_1^\dagger + \mathcal{O}_1^2 \phi_1 \phi_1^\dagger \right) + H_3 + H_4. \end{aligned} \quad (52)$$

In the RPA approach, the contribution of the interacting part of the Hamiltonian systematically transforms, in

the expressions of the various double commutators, the bare single-particle energy into the generalized mean-field single-particle energy (39) which is finite after mass renormalization in one spatial dimension:

$$\mathcal{O}_1^2 \rightarrow \varepsilon_1^2 = \mathcal{O}_1^2 + \frac{b}{2} \langle \phi^2 \rangle_R \quad (53)$$

where $\langle \phi^2 \rangle_R = (1/V) \sum \langle \phi_1 \phi_1^\dagger \rangle$ is in principle the correlated vacuum density (see subsect. 4.4) except in standard RPA where it is taken on the Gaussian ground state. This suggests to rewrite the Hamiltonian as

$$H = H_0 + H_{\text{int}} \quad (54)$$

with

$$H_{\text{int}} = H_3 + H_4 - \frac{b}{4} \langle \phi^2 \rangle_R \sum_1 : \phi_1 \phi_1^\dagger :_\varepsilon - V \frac{b}{8} \langle \phi^2 \rangle_\varepsilon^2. \quad (55)$$

$\langle \phi^2 \rangle_\varepsilon$ is the expectation value taken on the ground state of the mean-field quasi-particles with energies ε_1 and $: \dots :_\varepsilon$ is the normal ordering with respect to this vacuum, namely:

$$: \phi_1 \phi_1^\dagger :_\varepsilon = \phi_1 \phi_1^\dagger - \langle \phi_1 \phi_1^\dagger \rangle_\varepsilon. \quad (56)$$

We use again the notation $\langle \phi^2 \rangle_R$ for the scalar density. In renormalized RPA this scalar density is in principle the correlated one. In standard RPA the quantity $\langle \phi^2 \rangle_R$ appearing in H_{int} is identified with the mean-field scalar density $\langle \phi^2 \rangle_\varepsilon$ and ε refers to the Gaussian HFB mean field. In that case, the interacting Hamiltonian reduces to

$$(H_{\text{int}})_{\text{standard RPA}} = H_3 + : H_4 :_\varepsilon. \quad (57)$$

The H_0 Hamiltonian is obtained as $H - H_{\text{int}}$:

$$\begin{aligned} H_0 = & V \left(\frac{1}{2} \mu_0^2 s^2 + \frac{b}{24} s^4 \right) + \sum_1 \frac{1}{2} \left(\Pi_1 \Pi_1^\dagger + \mathcal{O}_1^2 \phi_1 \phi_1^\dagger \right) \\ & + \frac{b}{4} \langle \phi^2 \rangle_R \sum_1 : \phi_1 \phi_1^\dagger :_\varepsilon + V \frac{b}{8} \langle \phi^2 \rangle_\varepsilon^2. \end{aligned} \quad (58)$$

Again in the case of standard RPA the self-consistent scalar density $\langle \phi^2 \rangle_R$ is replaced in the above expression by the Gaussian HFB scalar density $\langle \phi^2 \rangle_\varepsilon$. H_0 can be rewritten as

$$H_0 = E_0 + \sum_1 \frac{1}{2} \left(: \Pi_1 \Pi_1^\dagger :_\varepsilon + \varepsilon_1^2 : \phi_1 \phi_1^\dagger :_\varepsilon \right). \quad (59)$$

It has the form of a free Hamiltonian for quasi-particles with mass m (see eq. (39)), *i.e.* $m^2 = \varepsilon_{\vec{q}}^2 - \vec{q}^2$. In one

spatial dimension, this mass is rendered finite by a simple mass renormalization:

$$m^2 = \mu^2 + \frac{b}{2} s^2 + \frac{b}{2} \left(\langle \phi^2 \rangle_R - \int_{-A}^{+A} \frac{dq}{2\pi} \frac{1}{2\sqrt{q^2 + \mu^2}} \right). \quad (60)$$

E_0 is the generalized mean-field vacuum energy

$$\begin{aligned} \frac{E_0}{V} &= \frac{1}{2} \mu_0^2 s^2 + \frac{b}{24} s^4 + \frac{b}{8} \langle \phi^2 \rangle_\varepsilon^2 \\ &+ \sum_1 \frac{1}{2} \left(\langle \Pi_1 \Pi_1^\dagger \rangle_\varepsilon + \mathcal{O}_1^2 \langle \phi_1 \phi_1^\dagger \rangle_\varepsilon \right) = \\ &\frac{1}{2} \mu_0^2 s^2 + \frac{b}{24} s^4 + \frac{1}{2V} \sum_{\vec{q}} \left(\frac{\varepsilon_q}{2} + \frac{\mathcal{O}_q^2}{2\varepsilon_q} \right) + \frac{b}{8} \langle \phi^2 \rangle_\varepsilon = \\ &\mu^2 \left[\frac{1}{2} s^2 + p s^4 + \frac{1}{8\pi} \left(\frac{m^2}{\mu^2} - 1 - \frac{m^2}{\mu^2} \ln \frac{m^2}{\mu^2} \right) \right. \\ &\quad \left. - \frac{3p}{16\pi^2} \left(\ln \frac{m^2}{\mu^2} \right)^2 \right]. \quad (61) \end{aligned}$$

We may notice that E_0/V is formally equal to the Gaussian energy density (12), (18). However, the single-particle energy ε_q and the corresponding quasi-particle mass m entering its expression now depends on the correlated scalar density $\langle \phi^2 \rangle$, in the self-consistent version.

$\langle H_3 \rangle$ and $\langle H_4 \rangle$ can be calculated once the RPA 2p-1p and 2p-2p GF are known (see the end of the previous subsection). But to calculate the total energy we also need the expectation value on the RPA ground state of the one-body operators $\phi_{\vec{q}} \phi_{\vec{q}}^\dagger$ and $\Pi_{\vec{q}} \Pi_{\vec{q}}^\dagger$ which are not directly given by the RPA calculation. This is the well-known difficulty of RPA, even in its simplest standard form, which frequently appears in the context of nuclear physics. In other words the calculation of the kinetic energy in RPA needs further manipulations. One possible way to achieve this is to use the so-called charging formula [11] for the calculation of the correlation energy (*i.e.* the deviation from the mean-field energy E_0) which has been historically introduced for the electron gas problem. Here we will show how to adapt the charging formula beyond the standard RPA, namely in the r-RPA case in the context of a quantum field theory.

The idea is to introduce a Hamiltonian where the coupling constant is varying between zero and its physical value. We thus define the auxiliary Hamiltonian

$$H'(\rho) = H_0 + \rho H_{\text{int}}, \quad H'(\rho = 1) = H. \quad (62)$$

The first thing to do is to solve the RPA problem for the $H'(\rho)$ Hamiltonian. For this purpose, one can notice that

its explicit form is given by

$$\begin{aligned} H'(\rho) &= V \left(\frac{1}{2} \mu_0^2 s^2 + \frac{b}{24} s^4 \right) \\ &+ \sum_1 \frac{1}{2} \left(\Pi_1 \Pi_1^\dagger + \mathcal{O}_1^2 \phi_1 \phi_1^\dagger \right) + \rho (H_3 + H_4) \\ &+ (1-\rho) \left(\frac{b}{4} \langle \phi^2 \rangle_R \sum_1 : \phi_1 \phi_1^\dagger :_\varepsilon + V \frac{b}{8} \langle \phi^2 \rangle_\varepsilon^2 \right). \quad (63) \end{aligned}$$

Up to constant terms, the Hamiltonian $H'(\rho)$ can be rewritten as

$$\begin{aligned} H'(\rho) &= V \left(\frac{1}{2} \mu_0^2 s^2 + \frac{b}{24} s^4 \right) + \sum_1 \left[\frac{1}{2} \Pi_1 \Pi_1^\dagger \right. \\ &\quad \left. + \frac{1}{2} \left(\mathcal{O}_1^2 + \frac{b}{2} (1-\rho) \langle \phi^2 \rangle_R \right) \phi_1 \phi_1^\dagger \right] + \rho (H_3 + H_4). \quad (64) \end{aligned}$$

For what concerns the solution of the $H'(\rho)$ RPA problem (in practice for the calculation of the commutators and double commutators entering the RPA equations) one has to make the following modifications with respect to the H problem:

$$\begin{aligned} H &\rightarrow H'(\rho), \\ \mathcal{O}_1^2 &\rightarrow \mathcal{O}_{1\rho}^2 = \mathcal{O}_1^2 + \frac{b}{2} (1-\rho) \langle \phi^2 \rangle_R, \\ H_3 + H_4 &\rightarrow \rho (H_3 + H_4). \quad (65) \end{aligned}$$

The single-particle energy occurring in the self-consistent RPA GF will be thus modified according to

$$\begin{aligned} \varepsilon_1^2 &\rightarrow \varepsilon_{1\rho}^2 = \mathcal{O}_{1\rho}^2 + \frac{b}{2} \rho \langle \phi^2 \rangle_{R\rho} = \\ &\mathcal{O}_1^2 + \frac{b}{2} ((1-\rho) \langle \phi^2 \rangle_R + \rho \langle \phi^2 \rangle_{R\rho}) = \\ &\varepsilon_1^2 + \frac{b}{2} \rho (\langle \phi^2 \rangle_{R\rho} - \langle \phi^2 \rangle_R), \quad (66) \end{aligned}$$

where $\langle \phi^2 \rangle_\rho$ is the scalar density in the correlated RPA ground state of $H'(\rho)$. Again the notation $\langle \phi^2 \rangle_{R\rho}$ is employed: in r-RPA, it coincides with the scalar density calculated on the self-consistent ground state of the $H'(\rho)$ Hamiltonian, while in the standard RPA, it coincides with the Gaussian density, *i.e.* calculated on the ground state of H_0 . The solution of the $H'(\rho)$ r-RPA problem is obtained formally from the solution of the H r-RPA problem (eq. (46)-(51)) by simply replacing ε_1 by $\varepsilon_{1\rho}$, the coupling constant b by ρb and $\mathcal{N}_1 = \langle \phi_1 \phi_1^\dagger \rangle_R$ by $\mathcal{N}_{1\rho} = \langle \phi_1 \phi_1^\dagger \rangle_{R\rho}$ calculated self-consistently with the $H'(\rho)$ Hamiltonian.

In the standard RPA all the expectation values of ϕ^2 are taken on the Gaussian ground state. In this case the energies ε_ρ remain identical to the Gaussian single-particle energies ε : standard RPA: $\langle \phi^2 \rangle_{R\rho} = \langle \phi^2 \rangle_R = \langle \phi^2 \rangle_\varepsilon$, $\varepsilon_{1\rho} = \varepsilon_1$.

Once the r-RPA problem is solved one can calculate the RPA ground-state energy relative to the starting Hamiltonian. Since both H_0 and H_{int} are independent of

ρ and since $H'(\rho = 1)$ coincides with the original H one can apply the charging formula. The RPA ground state energy can be obtained as

$$E_{\text{RPA}} = E_0 + \int_0^1 \frac{d\rho}{\rho} \langle \rho H_{\text{int}} \rangle_\rho, \quad (67)$$

where E_0 is the already calculated generalized mean-field energy. Using the Wick theorem with respect to the vacuum of quasi-particles with energies ε_ρ , the correlated part can be rewritten as

$$\begin{aligned} \langle \rho H_{\text{int}} \rangle_\rho &= \langle \rho H_3 \rangle_\rho + \langle \rho : H_4 :_{\varepsilon_\rho} \rangle_\rho \\ &- V \frac{\rho b}{8} (\langle \phi^2 \rangle_{\varepsilon_\rho} - \langle \phi^2 \rangle_\varepsilon)^2 - V \frac{\rho b}{4} (\langle \phi^2 \rangle_R - \langle \phi^2 \rangle_{\varepsilon_\rho}) \\ &\times \left(\frac{1}{V} \sum_1 \langle \phi_1 \phi_1^\dagger \rangle_\rho - \langle \phi^2 \rangle_\varepsilon \right). \end{aligned} \quad (68)$$

In this formula $\langle \phi^2 \rangle_R$ is as before the self-consistent scalar density of the original H . $\langle \phi^2 \rangle_\varepsilon$ is the scalar density on the generalized mean-field vacuum (vacuum of quasi-particles with energy ε_ρ) in the H problem and $\langle \phi^2 \rangle_{\varepsilon_\rho}$ corresponds to the equivalent quantity for the $H'(\rho)$ Hamiltonian. The remaining expectation values noted $\langle \dots \rangle_\rho$ have to be taken on the r-RPA ground state of $H'(\rho)$. The calculation of these latter expectation values are made using eqs. (46)-(51) where all the quantities are now relative to the $H'(\rho)$ problem as explained before.

In the particular case of the standard RPA the extra-term in the expression of the correlation energy disappears since, following eq. (57), one has

$$\langle \rho H_{\text{int}} \rangle_\rho = \langle \rho H_3 \rangle_\rho + \langle \rho : H_4 :_{\varepsilon} \rangle_\rho. \quad (69)$$

As we will see explicitly in the next subsection the expectation value of the normal-ordered Hamiltonian will involve an integration over the calculated Green's functions. In the case of the standard RPA the ρ integration can be done analytically. This is not the case in the r-RPA since these Green's functions will involve the ε_ρ 's and the self-consistent densities \mathcal{N}_ρ which depend explicitly on ρ .

4.3 Results in standard RPA in 1 + 1 dimension

Single-particle mode

The RPA single-particle mode ω_P with momentum \vec{P} is obtained as the solution of the equation see, *e.g.*, eq. (46):

$$\omega_P^2 = \varepsilon_P^2 + \Sigma(E = \omega_P, \vec{P}). \quad (70)$$

In the standard RPA, the densities are simply taken as $\mathcal{N}_1 = 1/2\varepsilon_1$, where the ε_1 's are the Gaussian single-particle energies. In that case $I(E, \vec{P})$ (eq. (48)) and consequently $\Sigma(E, \vec{P})$ (eq. (47)) are explicitly covariant in the sense that they depend only on $E^2 - \vec{P}^2$ and not on E and

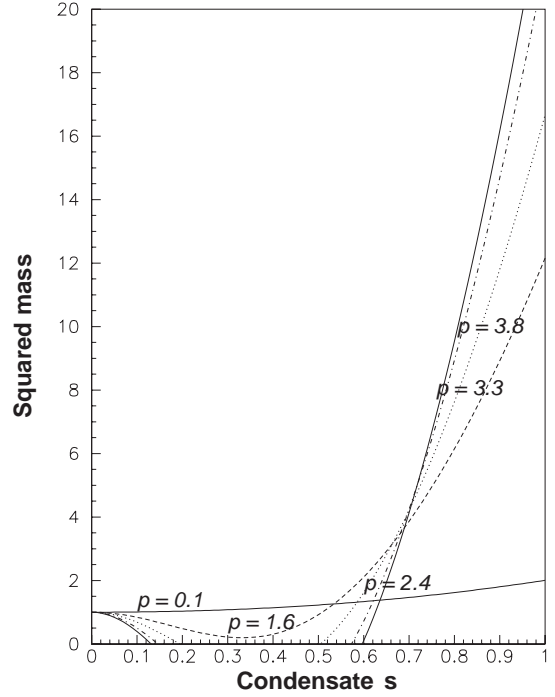


Fig. 3. Squared standard RPA mass (in unit of μ^2) as a function of s .

\vec{P} separately. After a simple boost-like change of variables one can show that

$$I(E, \vec{P}) \equiv I(E^2 - \vec{P}^2) = \int \frac{d\vec{t}}{(2\pi)^d} \frac{1}{\varepsilon_t} \frac{1}{E^2 - \vec{P}^2 - 4\varepsilon_t^2 + i\eta}. \quad (71)$$

Consequently, the RPA mode has a dispersion relation which is $\omega_P^2 = M^2 + \vec{P}^2$. The mass M of the single-particle RPA mode is thus the solution of the equation

$$M^2 = m^2 + \Sigma(E^2 - \vec{P}^2 = M^2). \quad (72)$$

In one spatial dimension the RPA mass operator is finite and there is no need of further coupling-constant renormalization. In fig. 3 the result of the calculation for various values of the dimensionless coupling constant p as a function of s . It is apparent that for p larger than a certain value the RPA equation may have an imaginary solution. Such a feature, which can appear in RPA, simply means that the HFB ground state is unstable. Hence for that particular theory one has to go to a superior version of the HFB-RPA approach.

Correlation energy

Although the standard RPA leads, in this particular theory, to an instability in a certain range of coupling constants, it is however interesting to look at the expression

of the correlation energy at least at a formal level. We divide the correlation energy density in three pieces

$$\frac{E_{\text{corr}}}{V} = \frac{E_{\text{corr}}^{(3)}}{V} + \frac{E_{\text{corr}}^{(4c)}}{V} + \frac{E_{\text{corr}}^{(4nc)}}{V}. \quad (73)$$

According to eqs. (50), (67), the piece corresponding to the three-body Hamiltonian reads

$$\begin{aligned} \frac{E_{\text{corr}}^{(3)}}{V} &= \int_0^1 \frac{d\rho}{\rho} \int \frac{d\vec{P}}{(2\pi)^d} \int \frac{i dE}{(2\pi)} e^{iE\eta^+} \frac{\rho^2 b^2 s^2}{6} \\ &\times \frac{I(E, \vec{P})}{1 - \frac{\rho b}{2} I(E, \vec{P})} \\ &\times \left(E^2 - \vec{P}^2 - m^2 - \frac{\rho^2 b^2 s^2}{2} \frac{I(E, \vec{P})}{1 - \frac{\rho b}{2} I(E, \vec{P})} \right)^{-1}. \end{aligned} \quad (74)$$

Notice that, as mentioned before, the ρ integration can be performed analytically. It is convenient to transform the energy integration into an integration on the imaginary axis (Wick rotation) by making the change of variable $E = iz$. In the usual RPA, the loop $I(E, \vec{P})$ and thus the whole integrand h_3 actually depends only on $E^2 - \vec{P}^2$. After the Wick rotation it depends only on $\mathcal{S} = z^2 + \vec{P}^2$ and the momentum integration can be done analytically in 1 + 1 dimension

$$\int_{-\infty}^{+\infty} \frac{dP}{2\pi} \int \frac{idE}{(2\pi)} e^{iE\eta^+} h_3(E^2 - \vec{P}^2) = - \int_0^\infty \frac{d\mathcal{S}}{4\pi} h_3(-\mathcal{S}). \quad (75)$$

For what concerns the four-body interacting piece we start from the explicit form of its normal ordering with respect to the ε basis

$$\begin{aligned} \langle : H_4 : \varepsilon \rangle &= \frac{b}{24 V \sqrt{\prod_i 2\varepsilon_i}} \delta_{1+2+3+4} \\ &\times \left(\left\langle b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger + b_{-1} b_{-2} b_{-3} b_{-4} + 4 b_1^\dagger b_2^\dagger b_3^\dagger b_{-4} \right. \right. \\ &\left. \left. + 4 b_1^\dagger b_{-2} b_{-3} b_{-4} + 6 b_1^\dagger b_2^\dagger b_{-3} b_{-4} \right\rangle \right), \end{aligned} \quad (76)$$

where summation over repeated indices is now omitted. As explained in subsect. 4.1, these matrix elements can be evaluated as an energy integral of two-particle Green's functions whose explicit expressions are given in the appendix. Noticing that all the matrix elements $\langle b^\dagger b^\dagger b^\dagger b \rangle$ are identically zero, it is convenient, after standard manipulations, to split $\langle : H_4 : \varepsilon \rangle$ into two pieces:

$$\langle : H_4 : \varepsilon \rangle = \langle : H_4 : \varepsilon \rangle^{(c)} + \langle : H_4 : \varepsilon \rangle^{(nc)},$$

$$\begin{aligned} \langle : H_4 : \varepsilon \rangle^{(c)} &= \frac{b}{24 V \sqrt{\prod_i 2\varepsilon_i}} \delta_{1+2+3+4} \\ &\times \left(\left\langle (b_1^\dagger b_2^\dagger + b_{-1} b_{-2}) (b_{-3} b_{-4} + b_3^\dagger b_4^\dagger) \right. \right. \\ &\left. \left. - 2 \delta_{1+3} \delta_{2+4} (1 + 2 \langle b_1^\dagger b_{11} \rangle) \right\rangle \right) = \\ &\frac{b}{24 V} \int \frac{i dE}{(2\pi)} e^{iE\eta^+} \left[\delta_{1+2-3-4} \right. \\ &\left. \times \left(G_{\phi_3\phi_4, \phi_1^\dagger\phi_2^\dagger}(E) - G_{\phi_3\phi_4, \phi_1^\dagger\phi_2^\dagger}^0 \right) \right], \\ \langle : H_4 : \varepsilon \rangle^{(nc)} &= \frac{b}{24 V \sqrt{\prod_i 2\varepsilon_i}} 4 \left\langle b_1^\dagger b_2^\dagger b_{-3} b_{-4} \right\rangle = \\ &\frac{b}{24 V} \int \frac{i dE}{(2\pi)} e^{iE\eta^+} \left[\delta_{1+2-3-4} G_{b_3 b_4, b_1^\dagger b_2^\dagger}(E) \right], \end{aligned} \quad (77)$$

where the suffices (c) and (nc) stand for covariant and non-covariant in a sense to be discussed just below. It is important to notice that the above result remains valid even in the case of the r-RPA, where the occupation numbers $\langle b^\dagger b \rangle$ do not vanish. Using the results of the appendix and the charging formula, the corresponding contributions to the correlation energy can now be obtained:

$$\begin{aligned} \frac{E_{\text{corr}}^{(4c)}}{V} &= \int_0^1 \frac{d\rho}{\rho} \int \frac{d\vec{P}}{(2\pi)^d} \\ &\times \int \frac{i dE}{(2\pi)} e^{iE\eta^+} I^2(E, \vec{P}) F(E, \vec{P}, \rho), \end{aligned} \quad (78)$$

$$\begin{aligned} \frac{E_{\text{corr}}^{(4nc)}}{V} &= \int_0^1 \frac{d\rho}{\rho} \int \frac{d\vec{P}}{(2\pi)^d} \\ &\times \int \frac{i dE}{(2\pi)} e^{iE\eta^+} I^{(1)2}(E, \vec{P}) F(E, \vec{P}, \rho), \end{aligned} \quad (79)$$

where the ρ integration can be again performed analytically, in standard RPA. $F(E, \vec{P}, \rho)$ has the explicit expression

$$\begin{aligned} F(E, \vec{P}, \rho) &\equiv F(E^2 - \vec{P}^2, \rho) = \\ &\frac{1}{24} \left(\frac{\rho^2 b^2}{1 - \frac{\rho b}{2} I(E, \vec{P})} + \frac{\rho^3 b^3 s^2}{(1 - \frac{\rho b}{2} I(E, \vec{P}))^2} \right. \\ &\left. \times \left(E^2 - \vec{P}^2 - m^2 - \frac{\rho^2 b^2 s^2}{2} \frac{I(E, \vec{P})}{1 - \frac{\rho b}{2} I(E, \vec{P})} \right)^{-1} \right). \end{aligned} \quad (80)$$

The contribution $E_{\text{corr}}^{(4c)}$ is explicitly covariant in the sense that the integrand depends only on $E^2 - \vec{P}^2$ and the trick of eq. (75) can be applied again. For what concerns $E_{\text{corr}}^{(4nc)}$, the explicit calculation is more delicate since

$$\begin{aligned} I^{(1)}(E, \vec{P}) &= \int \frac{d\vec{k}_1 d\vec{k}_2}{(2\pi)^d} \frac{\delta^{(d)}(\vec{P} - \vec{k}_1 - \vec{k}_2)}{2\varepsilon_1 2\varepsilon_2} \\ &\times \frac{1}{E - \varepsilon_1 + \varepsilon_2 + i\eta}, \end{aligned} \quad (81)$$

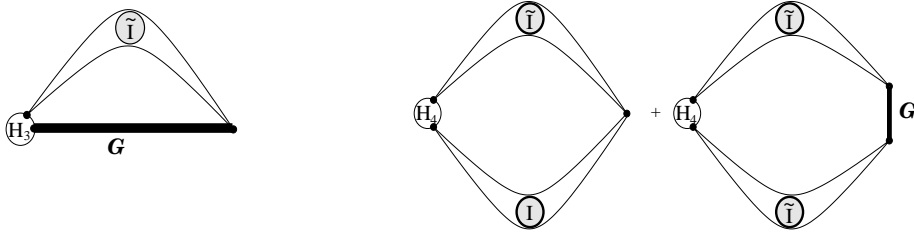


Fig. 4. Diagrammatic view of the correlation energy. In the actual calculation, the RPA one-particle propagator (thick line) is replaced by the mean-field one.

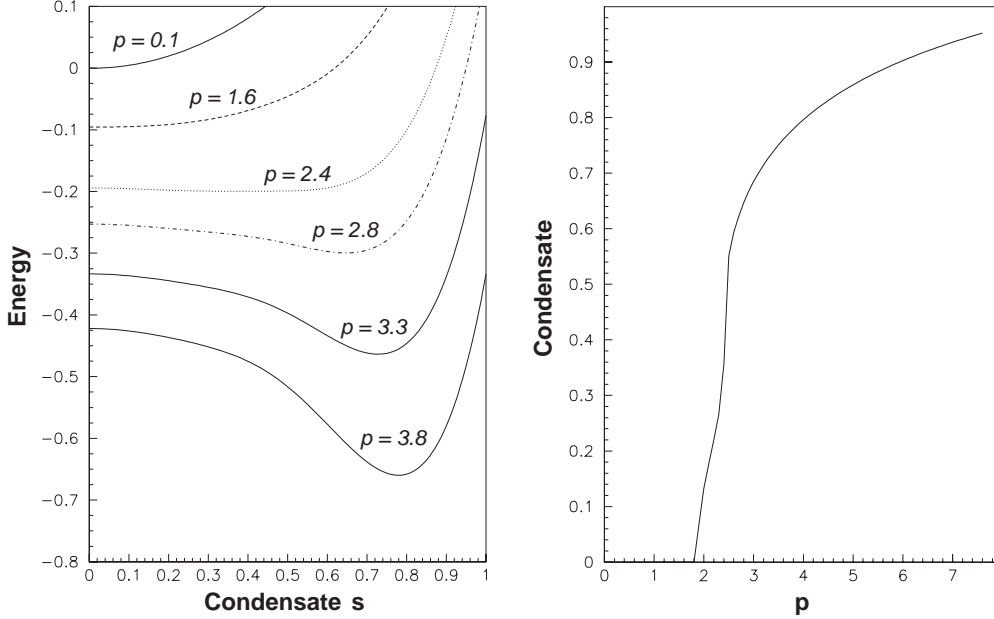


Fig. 5. Left panel: Effective potential for various values of the dimensionless coupling constant p in standard RPA. Right panel: value of the condensate minimizing the standard RPA effective potential as a function of p .

appearing in eq. (79), depends separately on E^2 and P^2 . However, as it is familiar in nuclear physics, $E_{\text{corr}}^{(4\text{nc})}$ which involves expectation values of the $b^\dagger b^\dagger b b$'s is of higher order in the Y amplitudes than $E_{\text{corr}}^{(4\text{c})}$ involving $bbbb$ ground-state matrix elements. Indeed, it can be checked analytically that to leading order in the interaction (*i.e.* replacing $F(E, \vec{P}, \rho)$ by a constant value $\rho^2 b^2 / 24$), $E_{\text{corr}}^{(4\text{nc})}$ identically vanishes.

Hence, we find that the correlation energy contains a piece, $E_{\text{corr}}^{(4\text{nc})}$, which is manifestly non-covariant even in the standard RPA. This problem has not been pointed out before, since, to our knowledge, the RPA correlation energy has never been calculated in a case of a relativistic theory for bosons. We have neglected this contribution in our preliminary numerical estimate for the reasons given just above. Nevertheless, we give the explicit result, involving a four-dimensional integration, of this non-covariant contribution. After some manipulations and change of integration variables, one obtains

$$\frac{E_{\text{corr}}^{(4\text{nc})}}{V} = \int_0^1 \frac{d\rho}{\rho} \int_0^\infty \frac{dS}{2\pi^2} \times \int_0^{\pi/2} d\theta \left(S \cos^2 \theta J^2(S, \theta) - I^2(-S) \right) F(-S), \quad (82)$$

with

$$J(S, \theta) = - \int \frac{dt}{2\pi} \frac{1}{\varepsilon_t \sqrt{4\varepsilon_t^2 + S \sin^2 \theta}} \frac{1}{S + 4\varepsilon_t^2} \quad (83)$$

to be compared with

$$I(-S) = - \int \frac{dt}{2\pi} \frac{1}{\varepsilon_t} \frac{1}{S + 4\varepsilon_t^2}. \quad (84)$$

As mentioned above there is an instability of the HFB ground state against RPA fluctuations which makes the correlation energy divergent. However, to have a first idea of the influence of the RPA fluctuations we replace in the above expressions for the correlation energy the RPA one-particle propagator by the mean-field one. This is illustrated in fig. 4 where a diagrammatic interpretation of the RPA correlation energy is shown. Adding this correlation energy to the mean-field energy E_0 one obtains the RPA effective potential, *i.e.* the RPA energy *versus* the condensate s for various values of the dimensionless coupling constant p . One gets a second-order phase transition with a critical coupling $p_c = 1.8$ (see fig. 5). This has to be compared with the lattice result [9] and cluster expansion technique [7] showing a second-order transition, respectively, at $p_c = 2.55$ and $p_c = 2.45$. It is fair to mention

that the neglected non-covariant contribution E_{corr}^{4c} is repulsive and will likely push the critical coupling constant to a higher value closer to the lattice result. Although this result is encouraging, it is obviously needed to go beyond the standard RPA to eliminate the unphysical instability mentioned above and seen in fig. 3.

4.4 Results in renormalized RPA in 1 + 1 dimension

Single-particle mode

In view of the calculation of the correlation energy we have now to solve the RPA problem for any value of ρ . The main problem is thus to determine the self-consistent $\mathcal{N}_{\vec{q},\rho} \equiv \langle \phi_{\vec{q}} \phi_{\vec{q}}^\dagger \rangle_\rho$. One very usual possibility is to calculate it self-consistently according to

$$\mathcal{N}_{\vec{P}\rho} = \int \frac{idE}{2\pi} e^{iE\eta^+} G_\rho(E, \vec{P}), \quad (85)$$

where $G_\rho(E, \vec{P})$ is the one-particle propagator for the $H'(\rho)$ problem. One serious difficulty is that covariance is now lost in the sense that the loop integral $I_\rho(E, \vec{P})$ and consequently the mass operator $\Sigma_\rho(E, \vec{P})$ depends separately on E and \vec{P} due to the presence of the density \mathcal{N} in its expression. This is certainly a weakness of the present approach. However, as discussed above, even standard RPA seems to have problems with covariance so we think that this additional difficulty simply reflects the fact that there is a general problem of RPA with respect to covariance. Further work is needed to clarify this point. On the other hand, one natural possibility to recover covariance consists in imposing that the correct $I_\rho(E, \vec{P})$ is obtained through its CM expression according to

$$I_\rho(E, \vec{P}) \equiv I_\rho(E^2 - \vec{P}^2) = \int \frac{dt}{2\pi} \frac{2\mathcal{N}_{t\rho}}{E^2 - \vec{P}^2 - 4\varepsilon_{t\rho}^2 + i\eta}. \quad (86)$$

Lets us call $\Omega_{P\rho} = \sqrt{M_\rho^2 + P^2}$ the RPA single-particle mode which is the solution of the equation

$$\Omega_{P\rho}^2 = \varepsilon_{P\rho}^2 + \Sigma_\rho(E^2 - \vec{P}^2 = M_\rho^2). \quad (87)$$

In the quasi-particle approximation, the solution for $\mathcal{N}_{P\rho}$ is

$$\mathcal{N}_{P\rho} \equiv \langle \phi_{\vec{P}} \phi_{\vec{P}}^\dagger \rangle_\rho = \frac{1}{2\Omega_{P\rho}}. \quad (88)$$

The self-consistent equation for the density thus becomes an equation for the mass of the RPA single-particle mode which explicitly reads

$$M_\rho^2 = m_\rho^2 + \frac{\rho^2 b^2 s^2}{2} \left(\frac{I_\rho}{1 - \frac{\rho b}{2} I_\rho} \right) (E^2 - P^2 = M_\rho^2) \quad (89)$$

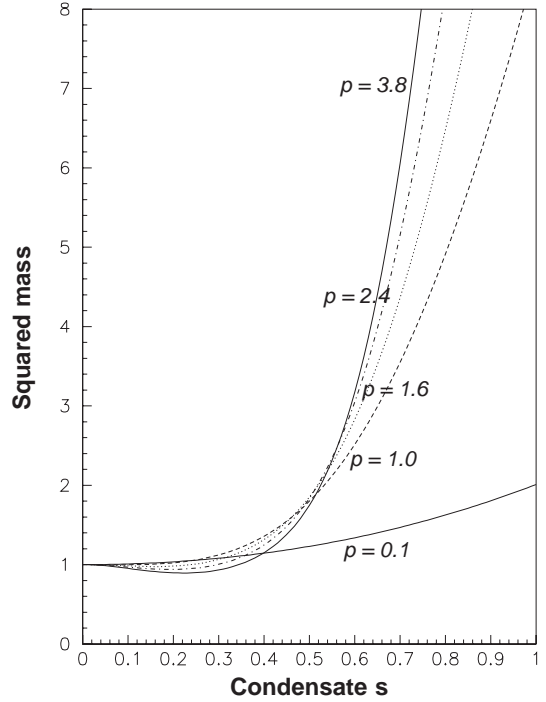


Fig. 6. Squared r-RPA mass as a function of s .

with the generalized mean-field single-particle mass given by

$$m_\rho^2 = \mu^2 + \frac{b}{2} s^2 + \frac{b}{2} \left(\langle \phi^2 \rangle - \int_{-A}^{+A} \frac{dq}{2\pi} \frac{1}{2\sqrt{q^2 + \mu^2}} \right) + \frac{\rho b}{2} (\langle \phi^2 \rangle_\rho - \langle \phi^2 \rangle), \quad (90)$$

which follows directly from (66). Hence we see that the equations for the generalized mean-field energy and for the single-particle RPA mode are now coupled, due to the presence of the self-consistent scalar density $\langle \phi^2 \rangle$. They finally reduce to determine M_ρ . The procedure to solve the resulting equation at a given value of b and s is the following. We first solve the equation for $\rho = 1$, which gives the RPA mode mass M , the densities \mathcal{N}_P , $\langle \phi^2 \rangle = \sum_P \mathcal{N}_P / V$ and the mean-field mass m . Once this is done we solve for M_ρ which allows to obtain m_ρ and $\varepsilon_{\rho P} = \sqrt{P^2 + m_\rho^2}$.

We show on fig. 6 the results of the calculation for the RPA mass M . We see that the instability problem has now disappeared. This is a first important success of the renormalized RPA.

Correlation energy

The calculation of the correlation energy can now be done by assuming again covariance in the sense explained just above. The results of eqs. (74), (75), (78), (79), (82) can be applied by just making in the final expressions, the replacements

$$I(-S) \rightarrow I_\rho(-S) = - \int \frac{dt}{2\pi} \frac{2\mathcal{N}_{t\rho}}{S + 4\varepsilon_{t\rho}^2}, \quad (91)$$

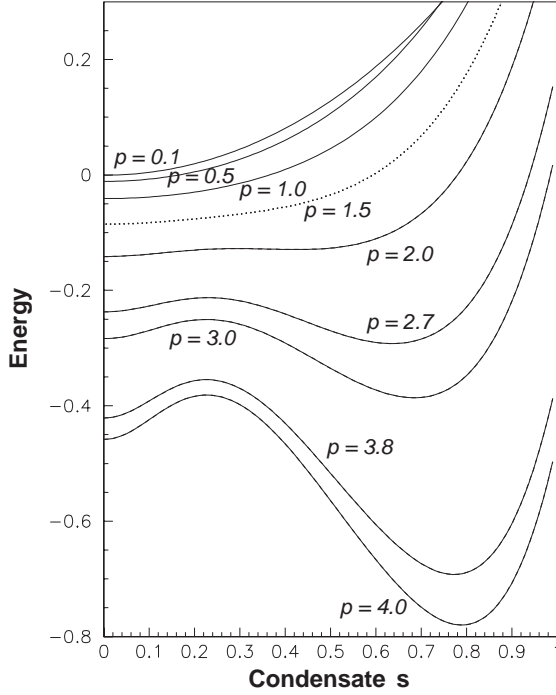


Fig. 7. Effective potential for various values of p in r-RPA.

$$J(\mathcal{S}, \theta) \rightarrow J_\rho(\mathcal{S}, \theta) = - \int \frac{dt}{2\pi} \frac{1}{\sqrt{4\varepsilon_{t\rho}^2 + \mathcal{S} \sin^2 \theta}} \times \frac{2\mathcal{N}_{t\rho}}{\mathcal{S} + 4\varepsilon_{t\rho}^2}. \quad (92)$$

In fig. 7, we show the effective potential for various values of the dimensionless coupling constant p , again neglecting the non-covariant piece (82) for the reasons explained above. For p below $\simeq 2$ there is only one minimum at $s = 0$, *i.e.* corresponding to a symmetry unbroken phase. Beyond this value, a weakly pronounced minimum starts to develop at finite s . The symmetry broken phase becomes stable at $p_c \simeq 2.3$ indicating a very weak first-order transition. It is satisfying to see that the value of p_c has moved in the right direction towards the value given by cluster expansion [7] and lattice calculation [9]. It remains to calculate the non-covariant contribution to see if it is able to transform this weak first-order transition into a genuine second-order one. Work in this direction is now in progress.

4.5 Towards full renormalized RPA

In the previous section we have presented a version of r-RPA which incorporates RPA correlations in the scalar density which are induced by the presence of a non-vanishing condensate. However, in the usual non-symmetry broken case this version is still equivalent to the standard RPA. To go beyond standard RPA is, as already stated, a difficult problem. One possibility is to introduce the dynamical mass operator modifying the single-particle

propagator. This mass operator is of the form

$$\Sigma_\alpha^{(d)}(t, t') = 2\varepsilon_\alpha \langle -iT([H, b_\alpha](t), [H, b_\alpha^\dagger](t')) \rangle_{\text{irr}} = \frac{1}{36} \sum_{123} \sum_{1'2'3'} V_{\alpha, -1, -2, -3} \times \langle -iT(\phi_1 \phi_2 \phi_3(t), \phi_1^\dagger \phi_2^\dagger \phi_3^\dagger(t')) \rangle V_{\alpha, -1', -2', -3'}. \quad (93)$$

Applying the perturbation theory on top of the already-calculated one-particle propagator, one obtains for the full propagator

$$G_\alpha^{\text{RPA}}(t, t') = G_\alpha(t, t') + \int dt_1 dt_2 G_\alpha(t, t_1) \Sigma_\alpha^{(d)}(t_1, t_2) G_\alpha(t_2, t'). \quad (94)$$

Using a factorization approximation, one obtains:

$$\Sigma_\alpha^{(d)}(t, t') = \frac{1}{6} \sum_{123} \sum_{1'2'3'} V_{\alpha, -1, -2, -3} \times \delta_{1,1'} G_1(t, t') G_{\phi_1 \phi_2, \phi_2^\dagger, \phi_3^\dagger}(t, t') V_{\alpha, -1', -2', -3'}. \quad (95)$$

The density \mathcal{N}_α can be in principle calculated as

$$\mathcal{N}_\alpha = \langle \phi_\alpha^\dagger \phi_\alpha \rangle \equiv i \lim_{t' \rightarrow t^+} G_\alpha^{\text{RPA}}(t, t') \equiv \int \frac{i dE}{2\pi} e^{iE\eta^+} G_\alpha^{\text{RPA}}(E). \quad (96)$$

The solution of this problem, *i.e.* to find self-consistently the scalar densities, is both formally and numerically very involved. However, as shown in a separate publication [19] at least in the symmetry-involved region of the anharmonic oscillator, this procedure reproduces to leading order in Y^2 the correct occupation number from the exact SCRPA ground-state wave function.

5 Further remarks and discussions on the formalism

It is well known in non-relativistic many-body theory that standard RPA corresponds to a bosonization either of pairs of fermion operators or pairs of boson operators. As a matter of fact, the RPA has first been invented for Fermi systems and the bosonization of boson pair operators has appeared much later in the literature [20,21]. However, in any case the bosonization of boson pair operators (which are themselves NOT ideal bosons) goes along the same lines as in the fermion case [22] and also the generalization to relativistic field theory [6] presents no particular obstacle (as a specific example, the bosonization technics has extensively been studied in the case of the NJL model, see, *e.g.*, [23]). In doing so, one would naturally replace the boson pair operators present in the definition of the Green's function (29) by ideal boson operators, that is for example $b_q^+ b_q^+ \rightarrow B_{qq^+}$, with B_{qq^+} being an ideal boson operator. With such a bosonization scheme the equation of motion for the Green's function

corresponding to eq. (29) would lead to an inhomogeneity \mathcal{N}_{ab} in the Dyson equation (32) which contains only unit operators on the diagonal. Inspection of eq. (32), however, reveals that \mathcal{N}_{ab} contains in addition one-body expectation values. This fact stems from the particular feature that in our approach we do not bosonise but stay with the original boson pair operators. Contrary to the usual two-body Green's function technique, the boson pair operators are, however, taken to be at equal times and not at two different times. This means that we single out the S -channel. It is not widely known that an exact integral equation for such two-body two-time Green's functions can be established analogous to the Dyson equation in the one-body case. The mass operator of this "two-body Dyson equation" presents, as in the one-body case, an instantaneous mean-field-type part and a truly dynamical part depending on the energy. Neglecting the latter leads precisely to the self-consistent equation (32). It has turned out that in this way, working with pairs of fermion or boson operators without bosonization, allows to better respect the Pauli principle and in the case of fermions this has given excellent results in a series of applications to non-trivial models where comparison with exact solutions was available [14, 20, 24]. There is no reason not to believe that this advantage should not carry over to the case of relativistic interacting boson fields. In the relativistic case, however, our ansatz (29) for the Green's function has the disadvantage that it is not manifestly covariant and this is certainly at the origin of some difficulties we encountered in this respect and which we have discussed in the main text. However, as we have shown in sect. 4, even standard RPA has, in our model at least, some difficulties with covariance. Further studies are necessary to elucidate this point and eventually to cure it. We believe nevertheless that the results we have found in the present study are encouraging and that we will be able to apply our theory in future work as successfully to relativistic field theory as we did already in the past for the non-relativistic many-body problem.

6 Conclusion

In this work we have tried to make the first step in elaborating an extension of the RPA theory, which has been very successful in the context of the non-relativistic many-body problem [13, 15–17, 24], to relativistic field theory. Applications of the standard RPA to the relativistic field theory has emerged in the recent past and proven its great potential interest [4–6]. The main quality of the RPA approach is to sum a certain class of diagrams (the rings) to all order and by the same token to restore spontaneously broken symmetries and to fulfill the conservation laws. The drawback of standard RPA is to ignore, at a certain step of its derivation, correlations in the vacuum (the quasi-boson approximation). This often entails a rather strong overbinding of the ground state. To avoid this approximation is the aim of the afore-mentioned extension of RPA leading to the so-called self-consistent RPA (SCRPA). An intermediate but considerably less compli-

cated version of this theory is the so-called renormalized RPA (r-RPA) where, with respect to standard RPA, only the occupation numbers are modified due to ground-state correlations. It is this latter version which we have tried to develop here in the context of relativistic field theory with application to the φ^4 theory in $1+1$ dimension. We have studied the transition to a symmetry broken phase in varying the coupling constant. We have found a very slight first-order phase transition and concluded that it will turn to second order, as it is expected in this model, once further correlations of the SCRPA are included. This opinion stems from the fact that going from the mean-field theory (Gaussian approximation) to the r-RPA solution the first-order character of the transition has been very much attenuated. We also point out a certain number of difficulties with the extension of RPA to relativistic field theory for bosons. This concerns for instance the fact that the approach is not manifestly covariant. Although the standard RPA yields at the end a covariant solution for the single-particle mode, surprisingly we found that it has difficulties for the calculation of the correlation energy with respect to covariance. Apparently this had not been noticed before. At the r-RPA level we find that covariance is violated already for the single-particle mode and we have to restore it by an *ad hoc* but natural assumption. It will be an interesting further study whether SCRPA inherently violates covariance or whether this is due to the approximations we have been forced to introduce. Another open question to be studied in the future concerns renormalization. In the present model study this difficulty was absent since the φ^4 theory in $1+1$ dimensions is super-renormalizable. However, in the general case, this problem has obviously to be mastered. Finally, a detailed comparison of the diagrammatic content of the RPA and the cluster expansion [7], in the context of relativistic bosonic theories with a broken symmetry, would be certainly of great interest.

In short we have applied for the first time an extension of the RPA theory, which turns out to be successful in the non-relativistic many-body problem, to a relativistic but schematic field-theoretic model. Although some problems are still present, we believe that our results are quite encouraging. Studies for the resolution of the remaining problems are under way.

We would like to thank Z. Aouissat for very useful discussions concerning various aspects of SCRPA and its application to field-theoretical problems.

Appendix A.

In this appendix we list the explicit expressions for the Green's functions. For this purpose we introduce various quantities:

$$I_{\beta\beta'}^{(1)}(E) = \frac{1}{2\varepsilon_\beta} \frac{1}{2\varepsilon_{\beta'}} \frac{\varepsilon_\beta \mathcal{N}_\beta + \varepsilon_{\beta'} \mathcal{N}_{\beta'}}{E - \varepsilon_\beta - \varepsilon_{\beta'} + i\eta},$$

$$I_{\beta\beta'}^{(2)}(E) = \frac{\varepsilon_\beta - \varepsilon_{\beta'}}{2\varepsilon_\beta \varepsilon_{\beta'}} \frac{\varepsilon_\beta \mathcal{N}_\beta - \varepsilon_{\beta'} \mathcal{N}_{\beta'}}{E^2 - (\varepsilon_\beta - \varepsilon_{\beta'})^2 + i\eta},$$

$$I_{\beta\beta'}^{(3)}(E) = -\frac{1}{2\varepsilon_\beta} \frac{1}{2\varepsilon_{\beta'}} \frac{\varepsilon_\beta \mathcal{N}_\beta + \varepsilon'_{\beta'} \mathcal{N}_{\beta'}}{E + \varepsilon_\beta + \varepsilon_{\beta'} - i\eta}. \quad (\text{A.1})$$

We also introduce the loop integrals

$$I_\alpha^{(i)}(E) = \frac{1}{V} \sum_{\beta\beta'} \delta_{\alpha-\beta-\beta'} I_{\beta\beta'}^{(i)}(E),$$

$$I_\alpha(E) = I_\alpha^{(1)}(E) + I_\alpha^{(2)}(E) + I_\alpha^{(3)}(E). \quad (\text{A.2})$$

In particular, for α corresponding to the momentum \vec{P} , one has the explicit expression

$$I(E, \vec{P}) \equiv I_{\alpha=\vec{P}}(E) = \int \frac{d\vec{k}_1 d\vec{k}_2}{(2\pi)^d} \delta^{(d)}(\vec{P} - \vec{k}_1 - \vec{k}_2)$$

$$\times \left[\frac{\varepsilon_1 + \varepsilon_2}{2\varepsilon_1 \varepsilon_2} \frac{\varepsilon_1 \mathcal{N}_1 + \varepsilon_2 \mathcal{N}_2}{E^2 - (\varepsilon_1 + \varepsilon_2)^2 + i\eta} \right.$$

$$\left. - \frac{\varepsilon_1 - \varepsilon_2}{2\varepsilon_1 \varepsilon_2} \frac{\varepsilon_1 \mathcal{N}_1 - \varepsilon_2 \mathcal{N}_2}{E^2 - (\varepsilon_1 - \varepsilon_2)^2 + i\eta} \right]. \quad (\text{A.3})$$

For the one-particle Green's functions one obtains

$$G_{\alpha\alpha'}(E) = \delta_{\alpha,\alpha'} \frac{E + \varepsilon_\alpha + \frac{\Sigma_\alpha(E)}{2\varepsilon_\alpha}}{2\varepsilon_\alpha} G_\alpha(E),$$

$$G_{-\alpha' - \alpha'}(E) = \delta_{\alpha,\alpha'} \frac{-E + \varepsilon_\alpha + \frac{\Sigma_\alpha(E)}{2\varepsilon_\alpha}}{2\varepsilon_\alpha} G_\alpha(E),$$

$$G_{-\alpha' \alpha'}(E) = G_{\alpha-\alpha'}(E) = \delta_{\alpha,\alpha'} \frac{\Sigma_\alpha(E)}{4\varepsilon_\alpha^2} G_\alpha(E), \quad (\text{A.4})$$

where the full propagator is

$$G_{\phi_\alpha \phi_\alpha^\dagger}(E) = \delta_{\alpha,\alpha'} G_\alpha(E) = \delta_{\alpha,\alpha'} (E^2 - \varepsilon_\alpha^2 - \Sigma_\alpha(E))^{-1}. \quad (\text{A.5})$$

The mass operator being given by

$$\Sigma_\alpha(E) = \frac{b^2 s^2}{2} \frac{I_\alpha(E)}{1 - \frac{b}{2} I_\alpha(E)}. \quad (\text{A.6})$$

For what concerns the 2p-1h and 2p-2p Green's functions we introduces indices i to label the destruction (creation) operators: $1 = \beta, \beta'(\beta^\dagger, \beta'^\dagger)$, $2 = (\beta, -\beta'^\dagger)_{\text{sym}}((\beta^\dagger, -\beta')_{\text{sym}})$ and $3 = -\beta^\dagger, -\beta'^\dagger(-\beta, -\beta')$. The results are:

$$G_{\beta\beta', \alpha^\dagger}^{(i)}(E) = G_{\alpha^\dagger, \beta\beta'}^{(i)}(E) =$$

$$\frac{b s}{\sqrt{V}} \delta_{\alpha-\beta-\beta'} \frac{I_{\beta\beta'}^{(i)}(E)}{1 - \frac{b}{2} I_\alpha(E)} \left(\frac{E + \varepsilon_\alpha}{2\varepsilon_\alpha} \right) G_{\phi_\alpha \phi_\alpha^\dagger}(E),$$

$$G_{\beta\beta', -\alpha}^{(i)}(E) = G_{-\alpha, \beta\beta'}^{(i)}(E) =$$

$$\frac{b s}{\sqrt{V}} \delta_{\alpha-\beta-\beta'} \frac{I_{\beta\beta'}^{(i)}(E)}{1 - \frac{b}{2} I_\alpha(E)} \left(\frac{-E + \varepsilon_\alpha}{2\varepsilon_\alpha} \right) G_{\phi_\alpha \phi_\alpha^\dagger}(E), \quad (\text{A.7})$$

$$G_{\beta\beta', \gamma\gamma'}^{(ij)}(E) = I_{\beta\beta'}^{(i)}(E) \delta_{i,j} (\delta_{\beta\gamma} \delta_{\beta'\gamma'} + \delta_{\beta\gamma'} \delta_{\beta'\gamma})$$

$$+ \frac{b}{V} \sum_\alpha \frac{\delta_{\alpha-\beta-\beta'} I_{\beta\beta'}^{(i)}(E) \delta_{\alpha-\gamma-\gamma'} I_{\gamma\gamma'}^{(i)}(E)}{1 - \frac{b}{2} I_\alpha(E)}$$

$$+ \frac{b^2 s^2}{V} \sum_\alpha \frac{\delta_{\alpha-\beta-\beta'} I_{\beta\beta'}^{(i)}(E) \delta_{\alpha-\gamma-\gamma'} I_{\gamma\gamma'}^{(i)}(E)}{(1 - \frac{b}{2} I_\alpha(E))^2} G_{\phi_\alpha \phi_\alpha^\dagger}(E). \quad (\text{A.8})$$

References

1. P. Ring, P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag, 1980).
2. P.M. Stevenson, Phys. Rev. D **30**, 1712 (1984); **32**, 1389 (1985); Y. Bihay, M. Consoli, Phys. Lett. B **157**, 48 (1985); P.M. Stevenson, Z. Phys. C **35**, 467 (1987); P.M. Stevenson, B. Allès, R. Tarrach, Phys. Rev. D **35**, 2407 (1987); A.K. Kerman, C. Martin, D. Vautherin, Phys. Rev. D **47**, 632 (1993); G. Amelino-Camelia, Phys. Lett. B **407**, 268 (1997) and references therein.
3. A.K. Kerman, D. Vautherin, Ann. Phys. (N.Y.) **192**, 408 (1989); C. Martin, D. Vautherin, hep-ph/9401261; I.I. Kogan, A. Kovner, Phys. Rev. D **52**, 3719 (1995) and references therein.
4. V. Dmitrasinovic, J.A. McNeil, J.R. Shepard, Z. Phys. C **69**, 35 (1996).
5. Z. Aouissat, G. Chanfray, P. Schuck, J. Wambach, Nucl. Phys. A **603**, 458 (1996).
6. Z. Aouissat, P. Schuck, J. Wambach, Nucl. Phys. A **618**, 402 (1997).
7. J.M. Håauser, W. Cassing, A. Peter, M.H. Thoma, Z. Phys. A **353**, 301 (1995).
8. W. Janke, T. Sauer, Nucl. Phys. Proc. Suppl. **42**, 917 (1995).
9. W. Loinaz, R.S. Willey, Phys. Rev. D **58**, 076003 (1998).
10. P.J. Marrero, E.A. Roura, D. Lee, Phys. Lett. B **471**, 45 (1999).
11. A.L. Fetter, J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New-York, 1971).
12. A.K. Kerman, Ann. Phys. (N.Y.) **269**, 55 (1998).
13. D.J. Rowe, Rev. Mod. Phys. **40**, 153 (1968); *Nuclear Collective Motion* (Methuen, London, 1970).
14. J. Dukelsky, P. Schuck, Nucl. Phys. A **512**, 466 (1990).
15. K. Hara, Progr. Theor. Phys. **32**, 88 (1964).
16. F. Catara, N. Dinh Dang, M. Sambatoro, Nucl. Phys. A **579**, 1 (1994).
17. J. Toivanen, J. Suhonen, Phys. Rev. Lett. **75**, 410 (1995).
18. P. Schuck, S. Ethofer, Nucl. Phys. A **212**, 269 (1973).
19. G. Chanfray, D. Davesne, M. Oertel, P. Schuck, in preparation.
20. P. Curutchet, J. Dukelsky, G.G. Dussel, A.J. Fendrik, Phys. Rev. C **40**, 2361 (1989).
21. R. Bijker, S. Pittel, J. Dukelsky, Phys. Lett. B **219**, 5 (1989).
22. J.P. Blaizot, G. Ripka, *Quantum Theory of Finite Systems* (Mit Press, 1986).
23. G. Ripka, *Quarks Bound by Chiral Fields. The Quark Structure of the Vacuum and of Light Mesons and Baryons* (Clarendon Press, 1997).
24. J. Dukelsky, P. Schuck, Mod. Phys. Lett. A **26**, 2429 (1991); P. Krüger, P. Schuck, Europhys. Lett. **72**, 395 (1994); J. Dukelsky, P. Schuck, Phys. Lett. B **387**, 233 (1996); J. Dukelsky, G. Röpke, P. Schuck, Nucl. Phys. A **628**, 17 (1998); J.G. Hirsch, A. Mariano, J. Dukelsky, P. Schuck, nucl-th/0109036.